

Energy from the gauge invariant observables

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Abstract

For a classical solution $|\Psi\rangle$ in Witten's cubic string field theory, the gauge invariant observable $\langle I|\mathcal{V}|\Psi\rangle$ is conjectured to be equal to the difference of the one-point functions of the closed string state corresponding to \mathcal{V} , between the trivial vacuum and the one described by $|\Psi\rangle$. If \mathcal{V} is taken to be the graviton vertex operator with vanishing momentum, the gauge invariant observable is expected to be proportional to the energy of $|\Psi\rangle$. We prove this relation assuming that $|\Psi\rangle$ satisfies equation of motion and some regularity conditions. We discuss how this relation can be applied to various solutions obtained recently.

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1 Introduction

A great variety of analytic classical solutions have been found for Witten's cubic string field theory [1], since the discovery of the analytic tachyon vacuum solution by Schnabl [2]¹. In order to study the physical properties of these solutions, important gauge invariant quantities to be calculated are the energy and the gauge invariant observables $\langle I|\mathcal{V}(i)|\Psi\rangle$ discovered in [4, 5]. The gauge invariant observable is conjectured to coincide with the difference of the one-point functions of an on-shell closed string state between the trivial vacuum and the one described by the solution $|\Psi\rangle$ [6, 7].

What we would like to show in this paper is that energy can be expressed by using a gauge invariant observable. Namely, for a static solution $|\Psi\rangle$ of the equation of motion, the gauge invariant observable with

$$\mathcal{V} = \frac{2}{\pi i} c\bar{c}\partial X^0\bar{\partial}X^0, \quad (1.1)$$

is proportional to the energy:

$$E = \frac{1}{g^2} \langle I|\mathcal{V}(i)|\Psi\rangle. \quad (1.2)$$

Here g is the coupling constant of the string field theory. Such a gauge invariant observable is the graviton one-point function and proportional to the expectation value of the energy momentum tensor $\langle T_{00}\rangle$. Therefore intuitively it is obvious that it coincides with the energy of the system. Usually, the energy is more difficult to calculate compared with the gauge invariant observables. For most of the solutions obtained so far, both the energy and the gauge invariant observable are calculated and it turns out that the results are consistent with (1.2).

In this paper, we will prove that (1.2) holds if $|\Psi\rangle$ satisfies the equation of motion and some regularity conditions. The state-operator correspondence of the worldsheet theory implies that the string field $|\Psi\rangle$ can be expressed as

$$\mathcal{O}_\Psi|0\rangle,$$

where $|0\rangle$ is the $\text{SL}(2,\mathbb{R})$ invariant vacuum and \mathcal{O}_Ψ can be expressed in terms of local operators on the upper half plane. We will first discuss the case in which \mathcal{O}_Ψ consists of local operators located away from the curve $|\xi| = 1$, where ξ is the complex coordinate on the upper half plane. As we will see, the proof of (1.2) is relatively easy in such a case. However, most of the solutions obtained since [2] do not satisfy this condition because they involve non-local operators such as K, B . Fortunately our method of proof can be refined to be applicable to such cases. We discuss applications of our results to the solutions obtained recently.

This paper is organized as follows. In section 2, we give a proof of the relation (1.2), assuming $|\Psi\rangle$ can be expressed using local operators. In section 3, we take Okawa type solutions [8, 9, 10] as an example and explain how we should generalize our method of proof to deal with solutions involving non-local operators K, B . In section 4 we apply our results to other solutions discovered recently. Section 5 is devoted to discussions.

¹For a review on these solutions, see [3].

2 A proof of (1.2) for local \mathcal{O}_Ψ

In this section, we consider the case in which \mathcal{O}_Ψ is made from local operators located away from $|\xi| = 1$.

2.1 Open string field theory in a weak gravitational background

In order to derive (1.2), we start from considering the following modification of the string field action,

$$S_h = -\frac{1}{g^2} \left[\frac{1}{2} \langle \Psi | Q | \Psi \rangle + \frac{1}{3} \langle \Psi | \Psi * \Psi \rangle + h \langle I | \mathcal{V}(i) | \Psi \rangle \right], \quad (2.1)$$

with $h \ll 1$. It has been shown in [11] that such a string field action describes string theory in a closed string background, for general on-shell \mathcal{V} . With \mathcal{V} in (1.1), the action should describe the open string field theory in a constant metric background.

By a general coordinate transformation, the constant metric can be turned into the original $\eta_{\mu\nu}$. Therefore we expect that we can somehow transform the string field action (2.1) into the original string field action. In order to do so, we notice that as an operator acting on $\mathcal{O}_\Psi |0\rangle$, \mathcal{V} can be expressed in a BRST exact form

$$\mathcal{V}(i) = \{Q, \chi\}, \quad (2.2)$$

where

$$\begin{aligned} \chi &\equiv \lim_{\delta \rightarrow 0} \left[\int_{P_1} \frac{d\xi}{2\pi i} j(\xi, \bar{\xi}) - \int_{\bar{P}_1} \frac{d\bar{\xi}}{2\pi i} \bar{j}(\xi, \bar{\xi}) - \kappa(e^{i\delta}, e^{-i\delta}) \right], \\ j(\xi, \bar{\xi}) &\equiv 4\partial X^0(\xi) \bar{c}\bar{\partial} X^0(\bar{\xi}), \\ \bar{j}(\xi, \bar{\xi}) &\equiv 4\bar{\partial} X^0(\bar{\xi}) c\partial X^0(\xi), \\ \kappa(\xi, \bar{\xi}) &\equiv \frac{1}{\pi i} (X^0(\xi, \bar{\xi}) - X^0(i, -i)) (c\partial X^0(\xi) - \bar{c}\bar{\partial} X^0(\bar{\xi})). \end{aligned}$$

Here P_1 is the path depicted in Fig 1 and along the arcs of the circle $|\xi| = 1$. Because of our assumption, the presence of \mathcal{O}_Ψ does not affect the operators defined on such contours. Because of the presence of $X^0(i, -i)$, $\kappa(\xi, \bar{\xi})$ is a well-defined operator on the worldsheet. Since j, \bar{j} diverge in the limit $\text{Im}\xi \rightarrow 0$, we have introduced $\delta > 0$ to regularize the divergence.

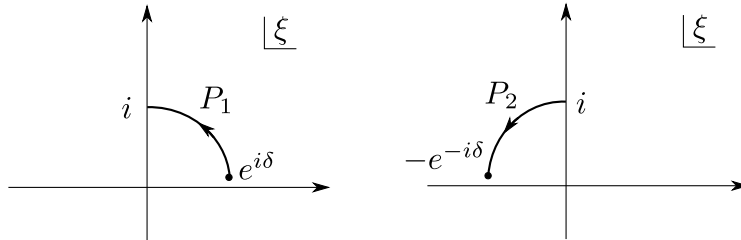


Figure 1: Contours P_1, P_2

(2.2) implies that in terms of the string field $|\Psi'\rangle$ defined as

$$|\Psi'\rangle \equiv |\Psi\rangle + h\chi |I\rangle ,$$

the string field action S_h is expressed as

$$S_h = -\frac{1}{g^2} \left[\frac{1}{2} \langle \Psi' | Q' | \Psi' \rangle + \frac{1}{3} \langle \Psi' | \Psi' * \Psi' \rangle \right] + \mathcal{O}(h^2) ,$$

with

$$Q' \equiv Q - h(\chi - \chi^\dagger) .$$

χ^\dagger denotes the BPZ conjugate of χ and

$$\begin{aligned} \chi - \chi^\dagger = \lim_{\delta \rightarrow 0} & \left[\int_{P_1+P_2} \frac{d\xi}{2\pi i} j(\xi, \bar{\xi}) - \int_{\bar{P}_1+\bar{P}_2} \frac{d\bar{\xi}}{2\pi i} \bar{j}(\xi, \bar{\xi}) \right. \\ & \left. - \kappa(e^{i\delta}, e^{-i\delta}) + \kappa(-e^{-i\delta}, -e^{i\delta}) \right] , \end{aligned}$$

where P_2 is the contour depicted in Fig. 1.

Therefore the string field theory in the weak gravitational background is given by the cubic action with the modified BRST operator Q' . This string field theory is similar to the one considered in [12] as the open string field theory in the soft dilaton background. They have shown that the effect of such a background corresponds to a rescaling of the string coupling constant g . It is straightforward to generalize the techniques of [12] to our case. Let us define

$$\begin{aligned} \mathcal{G} & \equiv \lim_{\delta \rightarrow 0} \left[\int_{P_1+P_2} \frac{d\xi}{2\pi i} g_\xi(\xi, \bar{\xi}) - \int_{\bar{P}_1+\bar{P}_2} \frac{d\bar{\xi}}{2\pi i} g_{\bar{\xi}}(\xi, \bar{\xi}) \right] , \\ g_\xi(\xi, \bar{\xi}) & \equiv 2(X^0(\xi, \bar{\xi}) - X^0(i, -i)) \partial X^0(\xi) , \\ g_{\bar{\xi}}(\xi, \bar{\xi}) & \equiv 2(X^0(\xi, \bar{\xi}) - X^0(i, -i)) \bar{\partial} X^0(\bar{\xi}) . \end{aligned} \tag{2.3}$$

Since $g_\xi, g_{\bar{\xi}}$ are singular at $\xi = i$, on the right hand side of (2.3) the integration contour is modified infinitesimally as in Figure 2. $g_\xi, g_{\bar{\xi}}$ are defined with the usual normal ordering prescription (B.2) and under a conformal transformation $\xi \rightarrow \xi'(\xi)$, g_ξ transforms as

$$g_{\xi'}(\xi', \bar{\xi}') = \frac{\partial \xi}{\partial \xi'} g_\xi(\xi, \bar{\xi}) + \frac{1}{2} \partial_{\xi'} \ln \frac{\partial \xi}{\partial \xi'} . \tag{2.4}$$

Using (2.4), one can deduce the following identities:

$$\mathcal{G} + \mathcal{G}^\dagger = 1 , \tag{2.5}$$

$$\langle \mathcal{G} \Psi_1 | \Psi_2 * \Psi_3 \rangle + \langle \Psi_1 | \mathcal{G} \Psi_2 * \Psi_3 \rangle + \langle \Psi_1 | \Psi_2 * \mathcal{G} \Psi_3 \rangle = \langle \Psi_1 | \Psi_2 * \Psi_3 \rangle , \tag{2.6}$$

$$[Q, \mathcal{G}] = \chi - \chi^\dagger . \tag{2.7}$$

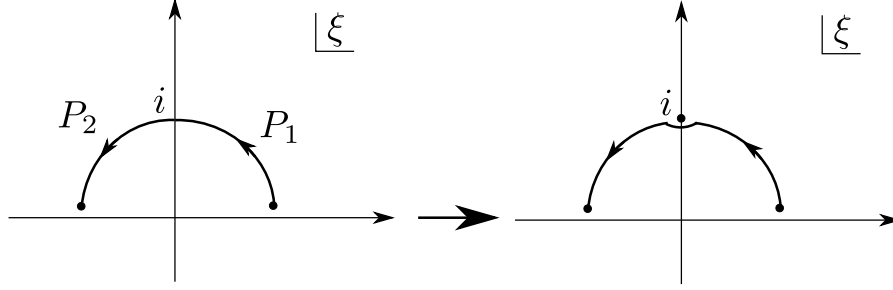


Figure 2: the contour to define \mathcal{G}

Then, in terms of

$$|\Psi''\rangle \equiv (1 - h\mathcal{G}) |\Psi'\rangle ,$$

S_h can be expressed as

$$S_h = -\frac{1+h}{g^2} \left[\frac{1}{2} \langle \Psi'' | Q | \Psi'' \rangle + \frac{1}{3} \langle \Psi'' | \Psi'' * \Psi'' \rangle \right] + \mathcal{O}(h^2) . \quad (2.8)$$

Thus S_h is proportional to the original string field theory action for the string field $|\Psi''\rangle$. By a field redefinition, the effect of the weak gravitational background is turned into a rescaling of the coupling constant g , which stems from the change of the spacetime volume. \mathcal{G} can be regarded as the generator of general coordinate transformation.

2.2 Derivation of (1.2)

We can derive (1.2) from the two expressions (2.1)(2.8) of S_h . Suppose that $|\Psi\rangle$ is a static solution of the equation of motion, and evaluate S_h using eqs.(2.1)(2.8). The right hand side of (2.1) can be expressed as

$$S_h = -E - \frac{h}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle ,$$

where $E = \frac{1}{g^2} \left[\frac{1}{2} \langle \Psi | Q | \Psi \rangle + \frac{1}{3} \langle \Psi | \Psi * \Psi \rangle \right]$ is the energy of the solution $|\Psi\rangle$. Since $|\Psi\rangle$ is a solution of the equation of motion, (2.8) can be given as

$$S_h = -(1+h)E + \mathcal{O}(h^2) .$$

Comparing these, we obtain

$$E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle . \quad (2.9)$$

There is a more direct way to derive (2.9), which is essentially equivalent to the one above and will be used in the subsequent sections. From (2.6), we can deduce the following expression of the energy $E = -\frac{1}{6g^2} \langle \Psi | \Psi * \Psi \rangle$:

$$E = -\frac{1}{2g^2} \langle \mathcal{G} \Psi | \Psi * \Psi \rangle . \quad (2.10)$$

The equation of motion implies

$$\begin{aligned}
\langle \mathcal{G}\Psi | \Psi * \Psi \rangle &= - \langle \mathcal{G}\Psi | Q | \Psi \rangle \\
&= - \langle [Q, \mathcal{G}] \Psi | \Psi \rangle - \langle \mathcal{G}Q\Psi | \Psi \rangle \\
&= - \langle \Psi | (\chi - \chi^\dagger) | \Psi \rangle + \langle \Psi * \Psi | \mathcal{G}^\dagger | \Psi \rangle .
\end{aligned} \tag{2.11}$$

Using (2.5), (2.6) and (2.2), we get

$$\begin{aligned}
\langle \Psi * \Psi | \mathcal{G}^\dagger | \Psi \rangle &= \langle \Psi * \Psi | \Psi \rangle - \langle \Psi * \Psi | \mathcal{G}\Psi \rangle \\
&= 2 \langle \mathcal{G}\Psi | \Psi * \Psi \rangle ,
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
\langle \Psi | (\chi - \chi^\dagger) | \Psi \rangle &= 2 \langle I | \chi | \Psi * \Psi \rangle \\
&= -2 \langle I | \mathcal{V}(i) | \Psi \rangle .
\end{aligned} \tag{2.13}$$

Substituting these into (2.10), we obtain

$$E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle . \tag{2.14}$$

Before closing this section, a few comments are in order:

- It is also possible to use \mathcal{V} for the graviton with spacelike polarizations and derive (1.2).
- Suppose that $|\Psi\rangle$ does not satisfy the equation of motion:

$$Q | \Psi \rangle + | \Psi * \Psi \rangle \equiv | \Gamma \rangle \neq 0 . \tag{2.15}$$

It is easy to see that the relation (2.14) is modified as

$$E = \frac{1}{g^2} \langle I | \mathcal{V}(i) | \Psi \rangle - \frac{1}{g^2} \langle I | \chi | \Gamma \rangle + \frac{1}{g^2} \langle \mathcal{G}\Psi | \Gamma \rangle . \tag{2.16}$$

3 Derivation of (1.2) for Okawa type solutions

Most of the nontrivial solutions obtained so far are described by using operators K, B . These operators are given as integrations of T, b along the contours which intersects $P_1, \bar{P}_1, P_2, \bar{P}_2$ and do not commute with $g_\xi, g_{\bar{\xi}}$ to be used to define \mathcal{G} . In order to prove (1.2) for such $|\Psi\rangle$, we need to define the quantities which appear in the previous section in the presence of such operators. Moreover it is not so straightforward to prove (2.11) in such a setup.

In this section, as a prototype of such solutions, we consider the Okawa type solutions [8, 9, 10]

$$\Psi = F(K) c \frac{KB}{1 - F(K)^2} c F(K) . \tag{3.1}$$

Here Ψ is expressed in terms of string fields K, B, c and the product of them is the star product². Ψ gives a solution of the equation of motion if $F(K), \frac{K}{1-F^2}$ are sufficiently regular functions of K . We will show that it is possible to define \mathcal{G} which acts on such solutions and prove (2.10)(2.11)(2.13) and derive (2.14).

It is assumed that $F(K), \frac{K}{1-F^2}$ are given in a Laplace transformed form

$$\begin{aligned} F(K) &= \int_0^\infty dL e^{-LK} f(L) , \\ \frac{K}{1-F^2} &= \int_0^\infty dL e^{-LK} \tilde{f}(L) . \end{aligned}$$

Substituting these into (3.1), we obtain an expression of Ψ

$$\Psi = \int_0^\infty dL e^{-LK} \psi(L) , \quad (3.2)$$

where

$$\begin{aligned} \psi(L) &= \int dL_1 dL_2 dL_3 \delta(L - L_1 - L_2 - L_3) \\ &\quad \times c(L_2 + L_3) Bc(L_3) f(L_1) \tilde{f}(L_2) f(L_3) , \end{aligned} \quad (3.3)$$

and

$$c(z) = e^{zK} c e^{-zK} . \quad (3.4)$$

Ψ can be considered as the Laplace transform of ψ . We express (3.2) as

$$\Psi = \mathcal{L} \{ \psi \} ,$$

where \mathcal{L} denotes the operation of the Laplace transform. Then $\psi(L)$ is expressed as

$$\psi(L) = \mathcal{L}^{-1} \{ \Psi \} (L) .$$

A few formulas concerning the Laplace transform to be used in the following are given in appendix A.

3.1 Definition of \mathcal{G}

Ψ is represented as a sum of wedge states with insertions $e^{-LK} \psi(L)$ as (3.2). In order to define \mathcal{G} which acts on such Ψ , the contour to be used should depend on the length L of the wedge state. So we introduce

$$\begin{aligned} \mathcal{G}(L, \Lambda, \delta) &\equiv \lim_{z_0 \rightarrow i\infty} \left[\int_{P_{L, \Lambda, \delta}} \frac{dz}{2\pi i} g_z(z, \bar{z}) - \int_{\bar{P}_{L, \Lambda, \delta}} \frac{d\bar{z}}{2\pi i} g_{\bar{z}}(z, \bar{z}) \right] , \\ g_z(z, \bar{z}) &= 2 (X^0(z, \bar{z}) - X^0(z_0, \bar{z}_0)) \partial X^0(z) , \\ g_{\bar{z}}(z, \bar{z}) &= 2 (X^0(z, \bar{z}) - X^0(z_0, \bar{z}_0)) \bar{\partial} X^0(z) , \end{aligned}$$

²See [13, 14] for details.

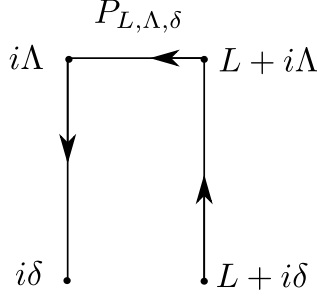


Figure 3: $P_{L, \Lambda, \delta}$

and define

$$\mathcal{G}\Psi \equiv \lim_{(\Lambda, \delta) \rightarrow (\infty, 0)} \int_0^\infty dL e^{-LK} \mathcal{G}(L, \Lambda, \delta) \psi(L) . \quad (3.5)$$

Here z is the cylinder coordinate which is related to the coordinate ξ used in (2.3) as

$$z = \frac{L}{2} + \frac{2L}{\pi} \arctan \xi .$$

The contour $P_{L, \Lambda, \delta}$ is the one depicted in Figure (3), which consists of straight lines.

The correlation functions of X^0 on the worldsheet are discussed in appendix B. We have introduced the reference point z_0 on the worldsheet in order to make $g_z, g_{\bar{z}}$ well-defined, which will be taken to be $i\infty$.

With \mathcal{G} thus defined, we will prove the identity (2.6) assuming Ψ_i ($i = 1, 2, 3$) does not involve the X^0 variable³. $\langle \mathcal{G}\Psi_1 | \Psi_2 * \Psi_3 \rangle$ is given as

$$\begin{aligned} & \lim_{(\Lambda, \delta) \rightarrow (\infty, 0)} \int_0^\infty dL_1 \int_0^\infty dL_2 \int_0^\infty dL_3 \\ & \times \langle e^{(L_2+L_3)K} \mathcal{G}(L_1, \Lambda, \delta) \psi_1(L_1) e^{-L_2K} \psi_2(L_2) e^{-L_3K} \psi_3(L_3) \rangle_{C_{L_1+L_2+L_3}} , \end{aligned} \quad (3.6)$$

in terms of the correlation function on the infinite cylinder $C_{L_1+L_2+L_3}$ with circumference $L_1 + L_2 + L_3$. Since ψ_i does not involve the X^0 variable, the correlation function on the right hand side of (3.6) is factorized as

$$\begin{aligned} & \langle e^{(L_2+L_3)K} \mathcal{G}(L_1, \Lambda, a) \psi_1(L_1) e^{-L_2K} \psi_2(L_2) e^{-L_3K} \psi_3(L_3) \rangle_{C_{L_1+L_2+L_3}} \\ & = \langle \mathcal{G}(L_1, \Lambda, a) \rangle_{C_{L_1+L_2+L_3}}^{X^0} \\ & \times \langle e^{(L_2+L_3)K} \psi_1(L_1) e^{-L_2K} \psi_2(L_2) e^{-L_3K} \psi_3(L_3) \rangle_{C_{L_1+L_2+L_3}} , \end{aligned} \quad (3.7)$$

where $\langle \dots \rangle_{C_{L_1+L_2+L_3}}^{X^0}$ denotes the correlation function with respect to X^0 variable on $C_{L_1+L_2+L_3}$. The expectation value $\langle \mathcal{G}(L_1, \Lambda, \delta) \rangle_{C_{L_1+L_2+L_3}}^{X^0}$ can be calculated using (B.3). In the limit

³(2.6) under such an assumption is enough for our purpose.

$\Lambda \rightarrow \infty, \delta \rightarrow 0$, we obtain

$$\lim_{(\Lambda, \delta) \rightarrow (\infty, 0)} \langle \mathcal{G}(L_1, \Lambda, \delta) \rangle_{C_{L_1+L_2+L_3}}^{X^0} = \frac{L_1}{L_1 + L_2 + L_3}. \quad (3.8)$$

Therefore we get

$$\begin{aligned} \langle \mathcal{G}\Psi_1 | \Psi_2 * \Psi_3 \rangle &= \int dL_1 dL_2 dL_3 \frac{L_1}{L_1 + L_2 + L_3} \\ &\quad \times \langle e^{(L_2+L_3)K} \psi_1(L_1) e^{-L_2K} \psi_2(L_2) e^{-L_3K} \psi_3(L_3) \rangle_{C_{L_1+L_2+L_3}}. \end{aligned}$$

In the same way, we obtain

$$\begin{aligned} \langle \Psi_1 | \mathcal{G}\Psi_2 * \Psi_3 \rangle &= \int dL_1 dL_2 dL_3 \frac{L_2}{L_1 + L_2 + L_3} \\ &\quad \times \langle e^{(L_2+L_3)K} \psi_1(L_1) e^{-L_2K} \psi_2(L_2) e^{-L_3K} \psi_3(L_3) \rangle_{C_{L_1+L_2+L_3}}, \\ \langle \Psi_1 | \Psi_2 * \mathcal{G}\Psi_3 \rangle &= \int dL_1 dL_2 dL_3 \frac{L_3}{L_1 + L_2 + L_3} \\ &\quad \times \langle e^{(L_2+L_3)K} \psi_1(L_1) e^{-L_2K} \psi_2(L_2) e^{-L_3K} \psi_3(L_3) \rangle_{C_{L_1+L_2+L_3}}, \end{aligned}$$

and from these (2.6) is obvious. (2.5) can also be proved in a similar way.

3.2 (2.11) for Okawa type solutions

Since (2.6) is satisfied, we get the expression (2.10) of the energy, which can be written as

$$E = \lim_{(\Lambda, \delta) \rightarrow (\infty, 0)} \frac{-1}{2g^2} \int dL_1 dL_2 \langle e^{L_2K} \mathcal{G}(L_1, \Lambda, \delta) \psi(L_1) e^{-L_2K} \mathcal{L}^{-1} \{ \Psi^2 \} (L_2) \rangle_{C_{L_1+L_2}},$$

in the setup of this section. We would like to show (2.11) in this setup. The equation of motion implies $\mathcal{L}^{-1} \{ Q\Psi + \Psi^2 \} = 0$. As is demonstrated in appendix A, $\mathcal{L}^{-1} \{ Q\Psi \}$ is different from $Q\mathcal{L}^{-1} \{ \Psi \}$ but expressed as (A.7), assuming $\alpha(\infty) = 0$ and $\alpha(0)$ is well-defined. Using (A.7), we get

$$\begin{aligned} &\lim_{(\Lambda, \delta) \rightarrow (\infty, 0)} \int dL_1 dL_2 \langle e^{L_2K} \mathcal{G}(L_1, \Lambda, \delta) \psi(L_1) e^{-L_2K} \mathcal{L}^{-1} \{ \Psi^2 \} (L_2) \rangle_{C_{L_1+L_2}} \\ &= - \lim_{(\Lambda, \delta) \rightarrow (\infty, 0)} \int dL_1 dL_2 \langle e^{L_2K} \mathcal{G}(L_1, \Lambda, \delta) \psi(L_1) e^{-L_2K} Q\psi(L_2) \rangle_{C_{L_1+L_2}} \\ &= \mathcal{A}_1 + \mathcal{A}_2, \end{aligned} \quad (3.9)$$

where

$$\mathcal{A}_1 \equiv - \lim_{(\Lambda, \delta) \rightarrow (\infty, 0)} \int dL_1 dL_2 \langle e^{L_2K} [Q, \mathcal{G}(L_1, \Lambda, \delta)] \psi(L_1) e^{-L_2K} \psi(L_2) \rangle_{C_{L_1+L_2}}, \quad (3.10)$$

$$\mathcal{A}_2 \equiv - \lim_{(\Lambda, \delta) \rightarrow (\infty, 0)} \int dL_1 dL_2 \langle e^{L_2K} \mathcal{G}(L_1, \Lambda, \delta) Q\psi(L_1) e^{-L_2K} \psi(L_2) \rangle_{C_{L_1+L_2}}. \quad (3.11)$$

Obviously $\mathcal{A}_1, \mathcal{A}_2$ correspond to the two terms $-\langle [Q, \mathcal{G}] \Psi | \Psi \rangle, -\langle \mathcal{G} Q \Psi | \Psi \rangle$ on the second line of (2.11), respectively. Therefore we expect that we can deduce

$$\mathcal{A}_1 \sim -\langle \Psi | (\chi - \chi^\dagger) | \Psi \rangle, \quad (3.12)$$

$$\mathcal{A}_2 \sim \langle \Psi * \Psi | \mathcal{G}^\dagger | \Psi \rangle. \quad (3.13)$$

As we will see, this is not the case.

Substituting (3.3) into (3.10), we obtain

$$\begin{aligned} \mathcal{A}_1 = & - \int dL_1 dL_2 \int dL'_1 dL'_2 dL'_3 \delta(L_1 - L'_1 - L'_2 - L'_3) \\ & \times f(L'_1) \tilde{f}(L'_2) f(L'_3) \\ & \times \langle e^{L_2 K} [Q, \mathcal{G}(L_1, \Lambda, \delta)] c(L'_2 + L'_3) Bc(L'_3) e^{-L_2 K} \psi(L_2) \rangle_{C_{L_1+L_2}}. \end{aligned} \quad (3.14)$$

The correlation function on the right hand side of (3.14) can be evaluated by plugging

$$\begin{aligned} [Q, \mathcal{G}(L, \Lambda, \delta)] = & \int_{P_{L, \Lambda, \delta}} \frac{dz}{2\pi i} 4\partial X^0(z) \bar{c} \bar{\partial} X^0(\bar{z}) - \int_{\bar{P}_{L, \Lambda, \delta}} \frac{d\bar{z}}{2\pi i} 4\bar{\partial} X^0(\bar{z}) c \partial X^0(z) \\ & - 2(c \partial X^0(i\infty) + \bar{c} \bar{\partial} X^0(-i\infty)) \left(\int_{P_{L, \Lambda, \delta}} \frac{dz}{2\pi i} \partial X^0(z) - \int_{\bar{P}_{L, \Lambda, \delta}} \frac{d\bar{z}}{2\pi i} \bar{\partial} X^0(\bar{z}) \right) \\ & + \int_{P_{L, \Lambda, \delta}} \frac{dz}{2\pi i} \frac{1}{2} \partial^2 c - \int_{\bar{P}_{L, \Lambda, \delta}} \frac{d\bar{z}}{2\pi i} \frac{1}{2} \bar{\partial}^2 \bar{c} \\ & + \int_{P_{L, \Lambda, \delta}} dz \partial \kappa(z, \bar{z}) + \int_{\bar{P}_{L, \Lambda, \delta}} d\bar{z} \bar{\partial} \kappa(z, \bar{z}), \quad (3.15) \\ \kappa(z, \bar{z}) = & \frac{1}{\pi i} (X^0(z, \bar{z}) - X^0(i\infty, -i\infty)) (c \partial X^0(z) - \bar{c} \bar{\partial} X^0(\bar{z})), \end{aligned}$$

into it and rewriting the result in terms of the operator formalism. We need to take into account the fact that the correlation functions are defined with time ordering with respect to the time variable $\text{Re} z$.

Since for $\text{Im} z, \text{Im} z' \sim \infty$,

$$\begin{aligned} \langle \partial X^0(z) \bar{\partial} X^0(\bar{z}') \rangle_{C_L} & \sim -2 \left(\frac{\pi}{L} \right)^2 \exp \left(\frac{2\pi i}{L} (z - \bar{z}') \right), \\ c(z) & \propto \exp \left(-\frac{2\pi i}{L} z \right), \end{aligned}$$

we can ignore the $\text{Im} z = \Lambda$ part of the contours $P_{L, \Lambda, \delta}, \bar{P}_{L, \Lambda, \delta}$ in the first and the second terms of (3.15), in the limit $\Lambda \rightarrow \infty$. One can see that the contributions from the terms on the second and the third lines of (3.15) vanish in the limit $\delta \rightarrow 0$, because of the boundary conditions of X^0, c, \bar{c} .

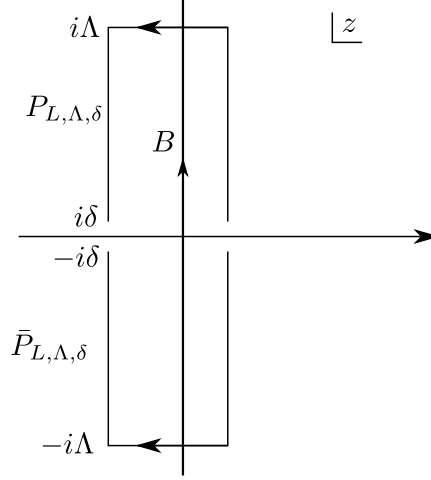


Figure 4: $P_{L,\Lambda,\delta}$ and B in \mathcal{A}_1

In calculating the contribution of the terms on the fourth line of (3.15), we need to be careful because the contours $P_{L,\Lambda,\delta}$, $\bar{P}_{L,\Lambda,\delta}$ intersect the contour for $B = \int_{a-i\infty}^{a+i\infty} \frac{dz}{2\pi i} b$ as depicted in Fig. 4. We obtain

$$\begin{aligned} & \left\langle e^{L_2 K} \left(\int_{P_{L,\Lambda,\delta}} dz \partial \kappa(z, \bar{z}) + \int_{\bar{P}_{L,\Lambda,\delta}} d\bar{z} \bar{\partial} \kappa(z, \bar{z}) \right) c(L'_2 + L'_3) B c(L'_3) e^{-L_2 K} \psi(L_2) \right\rangle_{C_{L_1+L_2}} \\ &= -\text{Tr} \left[e^{-L_1 K} c(L'_2 + L'_3) B c(L'_3) \kappa(i\delta, -i\delta) e^{-L_2 K} \psi(L_2) \right. \\ & \quad + e^{-L_1 K} \kappa(L_1 + i\delta, L_1 - i\delta) c(L'_2 + L'_3) B c(L'_3) e^{-L_2 K} \psi(L_2) \\ & \quad \left. + e^{-L_1 K} c(L'_2 + L'_3) \{B, \kappa(a + i\Lambda, a - i\Lambda)\} c(L'_3) e^{-L_2 K} \psi(L_2) \right]. \end{aligned}$$

Putting all these pieces together and taking the limit $\Lambda \rightarrow \infty$, we obtain

$$\begin{aligned} \mathcal{A}_1 &= - \int dL_1 dL_2 \text{Tr} \left[(\chi e^{-L_1 K} \psi(L_1) + e^{-L_1 K} \psi(L_1) \chi) e^{-L_2 K} \psi(L_2) \right] \\ & \quad - \int dL_1 dL_2 \frac{1}{L_1 + L_2} \text{Tr} \left[e^{-L_1 K} \alpha(L_1) e^{-L_2 K} \psi(L_2) \right], \end{aligned} \quad (3.16)$$

where $\alpha(L)$ is defined in (A.5) and χ here is given as

$$\begin{aligned} \chi &= \lim_{(\Lambda, \delta) \rightarrow (\infty, 0)} \left[\int_{i\delta}^{i\Lambda} \frac{dz}{2\pi i} 4\partial X^0(z) \bar{c} \bar{\partial} X^0(\bar{z}) \right. \\ & \quad - \int_{-i\delta}^{-i\Lambda} \frac{d\bar{z}}{2\pi i} 4\bar{\partial} X^0(\bar{z}) c \partial X^0(z) \\ & \quad \left. - \kappa(i\delta, -i\delta) \right]. \end{aligned}$$

The first term on the right hand side of (3.16) corresponds to $-\langle \Psi | (\chi - \chi^\dagger) | \Psi \rangle$ and we can see that (3.12) does not hold in the setup of this section.

\mathcal{A}_2 is evaluated by substituting (A.2) into the right hand side of (3.11). The correlation function in the integrand becomes

$$\begin{aligned} & \int dL'_1 dL'_2 dL'_3 \delta(L_1 - L'_1 - L'_2 - L'_3) f(L'_1) \tilde{f}(L'_2) f(L'_3) \\ & \times \left[\langle e^{L_2 K} \mathcal{G}(L_1, \Lambda, \delta) c \partial c(L'_2 + L'_3) B c(L'_3) e^{-L_2 K} \psi(L_2) \rangle_{C_{L_1+L_2}} \right. \\ & \quad - \langle e^{L_2 K} \mathcal{G}(L_1, \Lambda, \epsilon) c(L'_2 + L'_3) K c(L'_3) e^{-L_2 K} \psi(L_2) \rangle_{C_{L_1+L_2}} \\ & \quad \left. + \langle e^{L_2 K} \mathcal{G}(L_1, \Lambda, \epsilon) c(L'_2 + L'_3) B c \partial c(L'_3) e^{-L_2 K} \psi(L_2) \rangle_{C_{L_1+L_2}} \right]. \quad (3.17) \end{aligned}$$

The first and the third terms in the parenthesis are easily evaluated using (3.8). The second term in the parenthesis in (3.17) is calculated as

$$\begin{aligned} & \langle e^{L_2 K} \mathcal{G}(L_1, \Lambda, \delta) c(L'_2 + L'_3) K c(L'_3) e^{-L_2 K} \psi(L_2) \rangle_{C_{L_1+L_2}} \\ & = \text{Tr} [e^{-L_1 K} \mathcal{G}(L_1, \Lambda, \delta) c(L'_2 + L'_3) K c(L'_3) e^{-L_2 K} \psi(L_2)] \\ & = -\partial_t (\text{Tr} [e^{-L_1 K} \mathcal{G}(L_1, \Lambda, \delta) c(L'_2 + L'_3) e^{-tK} c(L'_3) e^{-L_2 K} \psi(L_2)]) \Big|_{t=0} \\ & = -\partial_t \left(\langle \mathcal{G}(L_1, \Lambda, \epsilon) \rangle_{C_{L_1+L_2+t}}^{X^0} \text{Tr} [e^{-L_1 K} c(L'_2 + L'_3) e^{-tK} c(L'_3) e^{-L_2 K} \psi(L_2)] \right) \Big|_{t=0}, \end{aligned}$$

and thus in the limit $\Lambda \rightarrow \infty$, we obtain

$$\begin{aligned} & \lim_{\Lambda \rightarrow \infty} \langle e^{L_2 K} \mathcal{G}(L_1, \Lambda, \delta) c(L'_2 + L'_3) K c(L'_3) e^{-L_2 K} \psi(L_2) \rangle_{C_{L_1+L_2}} \\ & = \frac{L_1}{L_1 + L_2} \langle e^{L_2 K} c(L'_2 + L'_3) K c(L'_3) e^{-L_2 K} \psi(L_2) \rangle_{C_{L_1+L_2}} \\ & \quad + \frac{L_1}{(L_1 + L_2)^2} \langle e^{L_2 K} c(L'_2 + L'_3) c(L'_3) e^{-L_2 K} \psi(L_2) \rangle_{C_{L_1+L_2}}. \end{aligned}$$

Putting these all together and using (A.7) and the equation of motion, we get

$$\begin{aligned} \mathcal{A}_2 &= \int dL_1 dL_2 \frac{L_1}{L_1 + L_2} \text{Tr} [e^{-L_1 K} \mathcal{L}^{-1} \{\Psi^2\}(L_1) e^{-L_2 K} \psi(L_2)] \\ & \quad + \int dL_1 dL_2 \frac{1}{L_1 + L_2} \text{Tr} [e^{-L_1 K} \alpha(L_1) e^{-L_2 K} \psi(L_2)]. \quad (3.18) \end{aligned}$$

Thus (3.13) does not hold either.

However, from (3.16) and (3.18), we get

$$\begin{aligned} & \lim_{(\Lambda, \delta) \rightarrow (\infty, 0)} \int dL_1 dL_2 \langle e^{L_2 K} \mathcal{G}(L_1, \Lambda, \delta) \psi(L_1) e^{-L_2 K} \mathcal{L}^{-1} \{\Psi^2\}(L_2) \rangle_{C_{L_1+L_2}} \\ & = - \int dL_1 dL_2 \text{Tr} [(\chi e^{-L_1 K} \psi(L_1) + e^{-L_1 K} \psi(L_1) \chi) e^{-L_2 K} \psi(L_2)] \\ & \quad + \int dL_1 dL_2 \frac{L_1}{L_1 + L_2} \text{Tr} [e^{-L_1 K} \mathcal{L}^{-1} \{\Psi^2\}(L_1) e^{-L_2 K} \psi(L_2)], \end{aligned}$$

which is exactly (2.11) in the setup in this section.

3.3 (1.2) for Okawa type solutions

(2.12) and (2.13) are easily shown using (A.7) and we obtain (1.2). Although the existence of B complicates the proof of (2.11), eqs.(2.10)~(2.13) hold with \mathcal{G} defined as (3.5).

In summary, we have proved (1.2) for Okawa type solutions Ψ assuming the following conditions:

- Ψ satisfies the equation of motion.
- $\alpha(\infty) = 0$ and $\alpha(0)$ is well-defined for $\alpha(L)$ defined in (A.5).

In addition to these, it is implicitly assumed that all the quantities which appear on the course of the calculations are finite⁴. Conditions other than the equation of motion are concerning the regularity of the solution. If the equation of motion is not satisfied, we obtain (2.16) with $|\Gamma\rangle$ given in (2.15).

4 Other solutions

We can use the method in the previous section and prove (1.2) for other types of solutions⁵. We will discuss BMT solution and Murata-Schnabl solution in the following.

4.1 BMT solution

In [17], Bonora, Maccaferri, and Tolla construct solutions corresponding to relevant deformations of BCFT, called BMT solution⁶. They enlarged the K, B, c algebra by adding a relevant matter operator ϕ which satisfies

$$\begin{aligned}\lim_{s \rightarrow 0} s\phi(s)\phi(0) &= 0, \\ [c, \phi] = [B, \phi] &= 0, \\ Q\phi &= c\partial\phi + \partial c\delta\phi.\end{aligned}$$

The BMT solution is given as

$$\Psi = c\phi - \frac{1}{K + \phi}(\phi - \delta\phi)Bc\partial c. \quad (4.1)$$

In order to realize the lump solution, ϕ is usually taken to be the so-called Witten deformation

$$\phi(s) = u \left(\frac{1}{2} : X^2 : (s) + \gamma - 1 + \ln(2\pi u) \right),$$

⁴This is also assumed in section 2.

⁵Our results will not be useful for the marginal deformation solutions, for which it is trivial to calculate the energy, but may be relevant [15] in the context of the discussions in Ref. [16].

⁶An earlier proposal for such solutions were made in [18]

or the cosine deformation

$$\phi(s) = u \left[-u^{-1/R^2} : \cos \left(\frac{1}{R} X \right) : (s) + A(R) \right] .$$

Here X direction is noncompact for Witten deformation and a circle of radius $R > \sqrt{2}$ for the cosine deformation. $A(R)$ is a constant determined in [17].

If one tries to define the $\frac{1}{K+\phi}$ which appear in the BMT solution as

$$\frac{1}{K+\phi} \equiv \int_0^\infty dt e^{-t(K+\phi)} ,$$

via the Schwinger parametrization, the integral on the right hand side diverges because $\lim_{t \rightarrow \infty} e^{-t(K+\phi)}$ coincides with the deformed sliver state $\tilde{\Omega}^\infty$. One way to regularize the divergence is to replace $\frac{1}{K+\phi}$ by $\frac{1}{K+\phi+\epsilon}$ with $1 \gg \epsilon > 0$ and consider

$$\Psi_\epsilon = c\phi - \frac{1}{K+\phi+\epsilon} (\phi - \delta\phi) Bc\partial c ,$$

but Ψ_ϵ suffers from an anomaly in equation of motion [19]:

$$Q\Psi_\epsilon + \Psi_\epsilon^2 = \Gamma_\epsilon \equiv \frac{\epsilon}{K+\phi+\epsilon} (\phi - \delta\phi) c\partial c .$$

In [20], the authors propose a way to deal with the problem using the distribution theory.

It is quite easy to compute the gauge invariant observables for the BMT solution, but it is much more difficult to calculate the energy. Our method can be used to improve the situation a bit. In [21, 19], the authors define a solution

$$\Psi'_\epsilon = c(\phi + \epsilon) - \frac{1}{K+\phi+\epsilon} (\phi - \delta\phi + \epsilon) Bc\partial c ,$$

as a possible regularization of the BMT solution, but it actually describe the tachyon vacuum. It is shown that if the energy of the solution Ψ'_ϵ is that of the tachyon vacuum, one can prove analytically that the energy of the BMT solution coincides with that of the lump solution [19, 21]. The gauge invariant observables of Ψ'_ϵ can be calculated analytically, which turn out to be equal to those of the tachyon vacuum but the energy is calculated only numerically [21, 22] in the case of the Witten deformation. We will use our method to calculate the energy of Ψ'_ϵ . It is quite straightforward to generalize the calculations in the previous section to Ψ'_ϵ , starting from the Laplace transformed form

$$\begin{aligned} \Psi'_\epsilon &= \int_0^\infty dL e^{-LK} \psi'_\epsilon(L) , \\ \psi'_\epsilon(L) &= \delta(L) c(\phi + \epsilon) - e^{-\epsilon L - \int_0^L ds \phi(s)} (\phi - \delta\phi + \epsilon) Bc\partial c , \end{aligned}$$

where

$$\phi(s) = e^{sK} \phi e^{-sK} ,$$

and the operators are time ordered. For Ψ'_ϵ , one can obtain

$$\mathcal{L}^{-1} \{Q\Psi'_\epsilon\} (L) = Q\mathcal{L}^{-1} \{\Psi'_\epsilon\} (L) - e^{LK} \partial_L (e^{-LK} \alpha'_\epsilon (L)) - \delta (L) \alpha'_\epsilon (0) ,$$

with

$$\alpha'_\epsilon (L) = e^{-\epsilon L - \int_0^L ds \phi(s)} (\phi - \delta\phi) c \partial c .$$

Since $\alpha'_\epsilon (\infty) = 0$ and $\alpha'_\epsilon (0)$ is well-defined, all the manipulations in the previous section are valid provided that we do not encounter any divergences on the course of the calculations. In the case of the Witten deformation, there exist divergences coming from noncompactness of the direction corresponding to X and our method is not applicable. Ψ'_ϵ corresponding to the cosine deformation does not seem to have such a problem⁷ and we can see that the energy coincides with that of the tachyon vacuum.

It may be possible to calculate the energy of Ψ_ϵ directly for the cosine deformation. Since Ψ_ϵ has an anomaly in equation of motion, we need to evaluate the second and the third terms of (2.16). In order to do so, we need to know the IR behavior of some correlation functions of ϕ .

4.2 Murata-Schnabl solution

Murata and Schnabl [24, 25] propose that the Okawa type solution (3.1) with

$$\begin{aligned} G(K) &\equiv 1 - F^2(K) \\ &= \left(\frac{K+1}{K} \right)^{N-1} , \end{aligned} \tag{4.2}$$

correspond to a configuration with N D-branes. Since the solution itself is singular for $N \neq 0, 1$, the authors need some regularization in calculating various quantities. In order to define the gauge invariant observables, they replace K by $K + \epsilon$ ($\epsilon \ll 1$) and consider

$$F(K + \epsilon) c \frac{B(K + \epsilon)}{1 - F^2(K + \epsilon)} c F(K + \epsilon) .$$

or its gauge equivalent

$$\Psi_\epsilon = F^2(K + \epsilon) c B \frac{K + \epsilon}{1 - F^2(K + \epsilon)} c .$$

The energy of the solution is calculated using a different way to regularize the divergence. They obtain the energy and the gauge invariant observables which coincide with those for N D-branes. It is necessary to find a more solid way to define the solution, and there are many attempts to rectify the situation [26, 27, 28, 29, 30].

⁷The partition function

$$g(uT) \equiv \text{Tr} e^{-T(K+\phi)} ,$$

can be calculated perturbatively [23] and is finite for $0 \leq uT < \infty$. The UV and IR behaviors of the correlation functions of ϕ 's are harmless.

As an application of our results, let us calculate the energy of Ψ_ϵ in this paper. Since Ψ_ϵ has an anomaly in equation of motion,

$$Q\Psi_\epsilon + \Psi_\epsilon^2 = \Gamma_\epsilon,$$

where

$$\begin{aligned}\Gamma_\epsilon &= \epsilon(1 - G_\epsilon(K)) c \frac{K + \epsilon}{G_\epsilon(K)} c, \\ G_\epsilon(K) &\equiv G(K + \epsilon),\end{aligned}$$

the relation we have is

$$E = \frac{1}{g^2} [\langle I | \mathcal{V}(i) | \Psi_\epsilon \rangle - \langle I | \chi | \Psi_\epsilon \rangle + \langle \mathcal{G} \Psi_\epsilon | \Gamma_\epsilon \rangle], \quad (4.3)$$

which can be proved as in the previous section. After some calculations, details of which are presented in appendix C, we obtain in the limit $\epsilon \rightarrow 0$

$$\begin{aligned}\langle I | \mathcal{V}(i) | \Psi_\epsilon \rangle &= \frac{N-1}{2\pi^2} \\ \langle I | \chi | \Psi_\epsilon \rangle &\rightarrow R_N, \\ \langle \mathcal{G} \Psi_\epsilon | \Gamma_\epsilon \rangle &\rightarrow 0,\end{aligned} \quad (4.4)$$

$$(4.5)$$

where

$$R_N \equiv \begin{cases} -\frac{i}{8\pi^3} \sum_{k=0}^{N-2} \frac{N!}{k!(k+2)!(N-2-k)!} \left((2\pi i)^{k+2} - (-2\pi i)^{k+2} \right) & , (N \geq 1), \\ \frac{i}{8\pi^3} \sum_{k=0}^{-N-1} \frac{(1-N)!}{k!(k+2)!(-N-1-k)!} \left((2\pi i)^{k+2} - (-2\pi i)^{k+2} \right) & , (N \leq 0). \end{cases}$$

Therefore we get the energy

$$E = \frac{1}{g^2} \left(\frac{N-1}{2\pi^2} - R_N \right).$$

This coincides with the desired value $\frac{N-1}{2\pi^2}$ for $N = -1, 0, 1, 2$. Thus, for these N , the anomaly Γ_ϵ is harmless at least in the calculation of energy, although we do not know the reason why this is so for $N = -1, 2$ ⁸.

⁸ $N = -1, 2$ may be argued to be special in the following sense. Ψ_ϵ is gauge equivalent to

$$\epsilon \left(\frac{1}{G_\epsilon} - 1 \right) c B G_\epsilon c,$$

which is regular in the limit $\epsilon \rightarrow 0$ for $N = -1, 2$. Another reason for $N = -1$ may be because there exists a regular solution [30].

5 Conclusion and discussion

In this paper, we present a way to show that the energy is proportional to a gauge invariant observable, which corresponds to the graviton one point function, for a classical solution in Witten's cubic open string field theory. We give a method which can be used to show this even for the solutions which involves K, B . Usually the gauge invariant observables are much easier to calculate compared with the energy. In a recent paper [31], it is found that the boundary states can also be constructed from the gauge invariant observables. Therefore now we possess a more efficient way to study the physical properties of solutions which has been or will be discovered.

Recently in [30] the authors propose several new types of solutions made from K, B, c . It seems that our method can be applied to these solutions and derive (1.2) if the solutions are sufficiently regular. One particularly interesting solution mentioned in [30] is the one due to Masuda, which is claimed to have the energy of the double brane configuration but the gauge invariant observables of the perturbative vacuum. It would be intriguing to check how our derivation of (1.2) fails for this solution.

Interrelationship between energy and the gauge invariant observable will be important in exploring various aspects of string fields. For example, in the case of the BMT solution, the calculation of gauge invariant observables reduces to the integral of total derivative. This implies that these gauge invariant observables may have some topological nature. On the other hand, in [27], the energy is interpreted to be the winding number in string field theory. Our results may shed some light on the study of the topological invariants of the space of string fields.

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A Laplace transformed form of the string field

We derive two formulas (A.1) (A.7) concerning the Laplace transform of the string field defined in section 3.

It is straightforward to generalize the definition of the Laplace transform and define $\tilde{A}(L)$ for any quantity A made from K, B, c . For two such quantities A_1, A_2 , it is easy to show

$$\mathcal{L}^{-1}\{A_1 A_2\}(L) = \int_0^L dL' e^{L'K} \tilde{A}_1(L - L') e^{-L'K} \tilde{A}_2(L') . \quad (\text{A.1})$$

The right hand side can be regarded as an operator version of convolution.

For $\psi(L)$ in (3.3),

$$\begin{aligned}
Q\psi(L) &= Q\mathcal{L}^{-1}\{\Psi\}(L) \\
&= \int dL_1 dL_2 dL_3 \delta(L - L_1 - L_2 - L_3) \\
&\quad \times [c\partial c(L_2 + L_3) Bc(L_3) - c(L_2 + L_3) Kc(L_3) + c(L_2 + L_3) Bc\partial c(L_3)] \\
&\quad \times f(L_1) \tilde{f}(L_2) f(L_3) , \tag{A.2}
\end{aligned}$$

which is not equal to

$$\begin{aligned}
\mathcal{L}^{-1}\{Q\Psi\}(L) &= \int dL_1 dL_2 dL_3 \delta(L - L_1 - L_2 - L_3) \\
&\quad \times \left[\{\partial c(L_2 + L_3) Bc(L_3) + c(L_2 + L_3) Bc\partial c(L_3)\} \right. \\
&\quad \times f(L_1) \tilde{f}(L_2) f(L_3) \\
&\quad \left. - c(L_2 + L_3) c(L_3) f(L_1) \mathcal{L}^{-1}\left\{\frac{K^2}{1-F^2}\right\}(L_2) f(L_3) \right] . \tag{A.3}
\end{aligned}$$

Therefore the BRST transformation and \mathcal{L}^{-1} do not commute with each other. Comparing (A.2) and (A.3), assuming $\alpha(0) = \alpha(\infty) = 0$, we obtain

$$\mathcal{L}^{-1}\{Q\Psi\}(L) = Q\mathcal{L}^{-1}\{\Psi\}(L) - e^{LK} \partial_L (e^{-LK} \alpha(L)) , \tag{A.4}$$

where

$$\alpha(L) \equiv \mathcal{L}^{-1}\left\{Fc\frac{K}{1-F^2}cF\right\}(L) . \tag{A.5}$$

We expect $\alpha(\infty) = 0$ for regular solutions. $\alpha(0)$ is related to the behavior of $F(K)$, $\frac{K}{1-F^2}$ for $K \sim \infty$ and may not vanish even if Ψ is regular. For example, the Erler-Schnabl solution [13] has

$$\begin{aligned}
f(L) &= \frac{1}{\Gamma(\frac{1}{2})} L^{-\frac{1}{2}} e^{-L} , \\
\alpha(L) &= e^{-L} \frac{1}{(\Gamma(\frac{1}{2}))^2} \int_0^L dL' (L - L')^{-\frac{1}{2}} L'^{-\frac{1}{2}} c\partial c(L') ,
\end{aligned}$$

and

$$\alpha(0) = c\partial c(0) ,$$

With $\alpha(0) \neq 0$, (A.4) cannot be valid for such solutions.

In order to get an identity similar to (A.4) for the solutions with $\alpha(\infty) = 0, \alpha(0) \neq 0$, we regularize Ψ and consider

$$\Psi_\eta \equiv F(K) e^{-\eta K} c \frac{BK}{1-F^2(K)} e^{-\eta K} cF(K) e^{-\eta K} ,$$

for $\eta > 0$. Ψ_η coincides with the original one in the limit $\eta \rightarrow 0$ and

$$\begin{aligned} \mathcal{L}^{-1}\{\Psi_\eta\}(L) &= \int dL_1 dL_2 dL_3 \delta(L - L_1 - L_2 - L_3) \\ &\quad \times c(L_2 + L_3) Bc(L_3) \mathcal{L}^{-1}\{F_\eta\}(L_1) \mathcal{L}^{-1}\{\tilde{F}_\eta\}(L_2) \mathcal{L}^{-1}\{F_\eta\}(L_3), \end{aligned}$$

where

$$\begin{aligned} F_\eta(K) &\equiv F(K) e^{-\eta K}, \\ \tilde{F}_\eta(K) &\equiv \frac{K}{1 - F^2(K)} e^{-\eta K}. \end{aligned}$$

$\mathcal{L}^{-1}\{F_\eta\}(L), \mathcal{L}^{-1}\{\tilde{F}_\eta\}(L)$ vanish for $L < \eta$ and we do not encounter any problem in deriving

$$\mathcal{L}^{-1}\{Q\Psi_\eta\}(L) = Q\mathcal{L}^{-1}\{\Psi_\eta\}(L) - e^{LK} \partial_L (e^{-LK} \alpha_\eta(L)), \quad (\text{A.6})$$

where

$$\alpha_\eta(L) \equiv \mathcal{L}^{-1}\{F_\eta c \tilde{F}_\eta c F_\eta\}(L).$$

$\alpha_\eta(L) \sim \alpha(L)$ for $L \gg \eta$ and $\alpha_\eta(L) = 0$ for $L < 3\eta$. Therefore, in the limit $\eta \rightarrow 0$,

$$\partial \alpha_\eta(L) \rightarrow \partial \alpha(L) + \delta(L) \alpha(0),$$

and (A.6) becomes

$$\mathcal{L}^{-1}\{Q\Psi\}(L) = Q\mathcal{L}^{-1}\{\Psi\}(L) - e^{LK} \partial_L (e^{-LK} \alpha(L)) - \delta(L) \alpha(0), \quad (\text{A.7})$$

which can be used for solutions with $\alpha(\infty) = 0, \alpha(0) \neq 0$, provided $\alpha(0)$ is well-defined. One can check that the Laplace transform of the right hand side yields $Q\Psi$.

B Correlation functions of X variables

In the calculations in section 3, we need the correlation functions of X variables on C_L . A conformal transformation which maps C_L to the upper half plane is given as

$$\begin{aligned} C_L &\rightarrow \text{UHP} \\ z &\rightarrow \xi = \tan \frac{\pi z}{L}. \end{aligned}$$

From the correlation functions

$$\begin{aligned} \langle \partial X^\mu(\xi) \partial X^\nu(\xi') \rangle_{\text{UHP}} &= \frac{-\frac{1}{2} \eta^{\mu\nu}}{(\xi - \xi')^2}, \\ \langle \partial X^\mu(\xi) \bar{\partial} X^\nu(\bar{\xi}') \rangle_{\text{UHP}} &= \frac{-\frac{1}{2} \eta^{\mu\nu}}{(\xi - \bar{\xi}')^2}, \end{aligned}$$

we can get

$$\begin{aligned}\langle \partial X^\mu(z) \partial X^\nu(z') \rangle_{C_L} &= -\frac{1}{2} \eta^{\mu\nu} \left(\frac{\pi}{L} \right)^2 \frac{1}{\sin^2 \frac{\pi(z-z')}{L}}, \\ \langle \partial X^\mu(z) \bar{\partial} X^\nu(\bar{z}') \rangle_{C_L} &= -\frac{1}{2} \eta^{\mu\nu} \left(\frac{\pi}{L} \right)^2 \frac{1}{\sin^2 \frac{\pi(z-\bar{z}')}{L}}.\end{aligned}\tag{B.1}$$

We are interested in the correlation function of the form $\langle (X^0(z, \bar{z}) - X^0(z_0, \bar{z}_0)) \partial X^0(z) \rangle_{C_L}$. Since the difference $X^0(z, \bar{z}) - X^0(z_0, \bar{z}_0)$ for some z_0, \bar{z}_0 can be written as

$$X^0(z, \bar{z}) - X^0(z_0, \bar{z}_0) = \int_{z_0}^z dz' \partial X^0(z') + \int_{\bar{z}_0}^{\bar{z}} d\bar{z}' \bar{\partial} X^0(\bar{z}'),$$

using $\partial X^0, \bar{\partial} X^0$, the correlation function $\langle (X^0(z, \bar{z}) - X^0(z_0, \bar{z}_0)) \partial X^0(z) \rangle_{C_L}$ is well-defined. Here it is assumed that the operators are normal ordered as

$$: X^0 \partial X^0 : (z, \bar{z}) \equiv \lim_{z' \rightarrow z} \left[X^0(z, \bar{z}) \partial X^0(z') - \frac{1}{2} \frac{1}{z' - z} \right]. \tag{B.2}$$

From (B.1) we obtain

$$\begin{aligned}\langle (X^0(z, \bar{z}) - X^0(z_0, \bar{z}_0)) \partial X^0(z) \rangle_{C_L} \\ = \frac{\pi}{2L} \left[\cot \frac{\pi(z - \bar{z})}{L} - \cot \frac{\pi(z - z_0)}{L} - \cot \frac{\pi(z - \bar{z}_0)}{L} \right].\end{aligned}\tag{B.3}$$

If one chooses the reference point z_0 to be $i\infty$, we get

$$\langle (X^0(z, \bar{z}) - X^0(i\infty, -i\infty)) \partial X^0(z) \rangle_{C_L} = \frac{\pi}{2L} \cot \frac{\pi(z - \bar{z})}{L}.$$

C Derivation of (4.4)(4.5)

We would like to calculate the second and the third terms on the right hand side of (4.3) in the limit $\epsilon \rightarrow 0$. These can be calculated basically using the s - z trick [24, 25].

Using

$$\mathcal{L}^{-1} \{ \Gamma_\epsilon \} (L) = \int_0^\infty dL_1 dL_2 \delta \left(L - \sum_i L_i \right) c(L_2) c(0) \mathcal{L}^{-1} \{ F_\epsilon^2 \} (L_1) \mathcal{L}^{-1} \left\{ \frac{K + \epsilon}{G_\epsilon} \right\} (L_2),$$

and

$$\begin{aligned}
& \langle c(L_2) c(0) c(z) \rangle_{C_L} \\
&= -\frac{1}{2} \left(\frac{L}{\pi} \right)^3 \left[\left(\sin \left(\frac{\pi z}{L} \right) \right)^2 \sin \frac{2\pi L_2}{L} - \left(\sin \left(\frac{\pi L_2}{L} \right) \right)^2 \sin \frac{\pi z}{L} \right], \\
& \left\langle c(L_2) c(0) \left(\int_{i\delta}^{i\Lambda} \frac{dz}{2\pi i} 4\partial X^0(z) \bar{c}\bar{\partial} X^0(\bar{z}) - \int_{-i\delta}^{-i\Lambda} \frac{d\bar{z}}{2\pi i} 4\bar{\partial} X^0(\bar{z}) c\partial X^0(z) \right) \right\rangle_{C_L} \\
& \xrightarrow{(\delta, \Lambda) \rightarrow (0, \infty)} \frac{1}{4\pi} \left(\frac{L}{\pi} \right)^2 \sin \frac{2\pi L_2}{L}, \\
& \langle c(L_2) c(0) \kappa(i\delta, -i\delta) \rangle_{C_L} \xrightarrow{\delta \rightarrow 0} 0,
\end{aligned}$$

$\langle I | \chi | \Psi_\epsilon \rangle$ becomes

$$\begin{aligned}
\langle I | \chi | \Psi_\epsilon \rangle &= \frac{-1}{4\pi^3} \epsilon \int_0^\infty ds s^2 \int_0^\infty dL_1 dL_2 \delta \left(s - \sum_i L_i \right) \\
& \quad \times \mathcal{L}^{-1} \{ G_\epsilon \} (L_1) \mathcal{L}^{-1} \left\{ \frac{K + \epsilon}{G_\epsilon} \right\} (L_2) \sin \frac{2\pi}{s} L_2 \\
&= \frac{-1}{4\pi^3} \epsilon \int_0^\infty ds s^2 \int_0^\infty dL_1 dL_2 \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{(s - \sum_i L_i)z} \\
& \quad \times \mathcal{L}^{-1} \{ G_\epsilon \} (L_1) \mathcal{L}^{-1} \left\{ \frac{K + \epsilon}{G_\epsilon} \right\} (L_2) \sin \frac{2\pi}{s} L_2 \\
&= \frac{i}{8\pi^3} \epsilon \int_0^\infty ds s^2 \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{sz} G_\epsilon(z) \Delta \left(\frac{z + \epsilon}{G_\epsilon} \right) \\
&= \frac{i}{8\pi^3} \epsilon \int_0^\infty ds s^2 \oint_P \frac{dz}{2\pi i} e^{sz} G_\epsilon(z) \Delta \left(\frac{z + \epsilon}{G_\epsilon} \right). \tag{C.1}
\end{aligned}$$

Here P is contour on the z plane shown in figure 5 and Δ is defined as [24, 25]

$$\Delta F(z) = F \left(z - \frac{2\pi i}{s} \right) - F \left(z + \frac{2\pi i}{s} \right).$$

For the Murata-Schnabl solution (4.2), (C.1) is evaluated as

$$\begin{aligned}
\langle I | \chi | \Psi_\epsilon \rangle &= R_N + \mathcal{O}(\epsilon), \tag{C.2} \\
R_N &\equiv \begin{cases} -\frac{i}{8\pi^3} \sum_{k=0}^{N-2} \frac{N!}{k!(k+2)!(N-2-k)!} \left((2\pi i)^{k+2} - (-2\pi i)^{k+2} \right) & , (N \geq 1), \\ \frac{i}{8\pi^3} \sum_{k=0}^{-N-1} \frac{(1-N)!}{k!(k+2)!(-N-1-k)!} \left((2\pi i)^{k+2} - (-2\pi i)^{k+2} \right) & , (N \leq 0), \end{cases}
\end{aligned}$$

for $\epsilon \ll 1$.

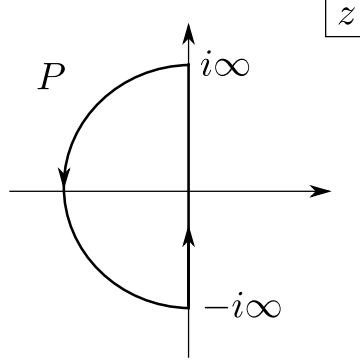


Figure 5: contour P

The third term on the right hand side of (4.3) becomes

$$\begin{aligned}
& \int dL_1 dL_2 \frac{L_1}{L_1 + L_2} \langle e^{L_2 K} \mathcal{L}^{-1} \{ \Psi_\epsilon \} (L_1) e^{-L_2 K} \mathcal{L}^{-1} \{ \Gamma_\epsilon \} (L_2) \rangle_{C_{L_1+L_2}} \\
&= \epsilon \int_0^\infty ds \prod_{i=1}^4 dL_i \delta \left(s - \sum_{i=1}^4 L_i \right) \frac{L_1 + L_2}{s} \\
&\quad \times \text{Tr} \left[e^{-L_1 K} \mathcal{L}^{-1} \{ G_\epsilon \} (L_1) c B e^{-L_2 K} \mathcal{L}^{-1} \left\{ \frac{K + \epsilon}{G_\epsilon} \right\} (L_2) c \right. \\
&\quad \left. \times e^{-L_3 K} \mathcal{L}^{-1} \{ G_\epsilon \} (L_3) c e^{-L_4 K} \mathcal{L}^{-1} \left\{ \frac{K + \epsilon}{G_\epsilon} \right\} (L_4) c \right].
\end{aligned}$$

Using

$$L \mathcal{L}^{-1} \{ f \} (L) = \mathcal{L}^{-1} \{ \partial f \} (L),$$

and eq.(2.5) in [25], we obtain

$$\begin{aligned}
& \int dL_1 dL_2 \frac{L_1}{L_1 + L_2} \langle e^{L_2 K} \mathcal{L}^{-1} \{ \Psi_\epsilon \} (L_1) e^{-L_2 K} \mathcal{L}^{-1} \{ \Gamma_\epsilon \} (L_2) \rangle_{C_{L_1+L_2}} \\
&= \frac{i}{8\pi^3} \epsilon \int_0^\infty ds s \oint_C \frac{dz}{2\pi i} e^{sz} \frac{1}{2i} \\
&\quad \times \left\{ \left[\frac{z + \epsilon}{G_\epsilon}, G_\epsilon, \frac{z + \epsilon}{G_\epsilon}, G'_\epsilon \right] + \left[\left(\frac{z + \epsilon}{G_\epsilon} \right)', G_\epsilon, \frac{z + \epsilon}{G_\epsilon}, G_\epsilon \right] \right\},
\end{aligned}$$

where

$$\begin{aligned}
[F_1, F_2, F_3, F_4] \equiv & [-F_1 \Delta F_2 F_3 F'_4 + F_1 \Delta (F_2 F'_3) F_4 + F_1 \Delta (F_2 F_3) F'_4 - F_1 F'_2 F_3 \Delta F_4 \\
& + F_1 F'_2 \Delta (F_3 F_4) + F_1 F_2 \Delta (F'_3 F_4) - F_1 \Delta (F_2 F'_3 F_4) - F_1 (F_2 \Delta F_3 F_4)'] .
\end{aligned}$$

The contribution of $\mathcal{O}(\epsilon^0)$ is given by the following replacements

$$\begin{aligned} G'(z) &\rightarrow -(N-1)G(z) , \\ G''(z) &\rightarrow N(N-1)\frac{1}{z^2}G(z) , \\ \left(\frac{z}{G}\right)'(z) &\rightarrow NG^{-1}(z) , \end{aligned}$$

and one can see

$$\int dL_1 dL_2 \frac{L_1}{L_1 + L_2} \langle e^{L_2 K} \mathcal{L}^{-1} \{\Psi_\epsilon\}(L_1) e^{-L_2 K} \mathcal{L}^{-1} \{\Gamma_\epsilon\}(L_2) \rangle_{C_{L_1+L_2}} \sim \mathcal{O}(\epsilon) .$$

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